Moore Space Homotopy Groups

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For an abelian group A a Moore space $M(A, k)$ is a space that has *homology* groups A in degree k and *reduce homology groups* zero everywhere else [Hat02, Example 2.40]. For $n > 1$ by [Hat02, Example 4.34] there is a unique *simply connected* Moore space up to homotopy for any given abelian group, for this reason it may be included in the definition that the Moore space be simply connected. In general $M(A, k) = \Sigma M(A, k-1)$. We may denote the Moore space valued homotopy groups

$$\pi_k(X; G) \quad := \quad [M(G, k-1), X]_*$$

These are alternatively called homotopy groups with coefficients. There is a universal coefficients for these groups [Hat02, Prop 4H.2], for $n > 1$

$$0 \to \text{Ext}(G, \pi_{n+1}(X)) \to \pi_{n+1}(X; G) \to \text{Hom}(G, \pi_n(X)) \to 0$$

and the dual version for Tor and tensor.

Example. *It is well known that S^n is a $M(\mathbb{Z}, n)$, however S^n is not simply connected. One can see that uniqueness fails through the standard homology sphere examples.*

Let $A = \mathbb{Z}_p$. Then the Moore space $M(A, n)$ is S^n with a D^{n+1} disc glued to it along the discs boundary by a degree p map.

This is [Hat02, Thm 4.58]To compare the Moore space homotopy groups with regular homotopy groups for this example $A = \mathbb{Z}_p$ we can look at the Puppe sequence for the cofibration

$$S^n \hookrightarrow S^n \cup_p D^{n+1} = M(\mathbb{Z}_p, n)$$

That sequence looks like

$$S^n \to M(\mathbb{Z}_p, n) \xrightarrow{\text{collapse}} M(\mathbb{Z}_p, n)/S^n = S^{n+1} \to \Sigma S^n \to \Sigma M(\mathbb{Z}_p, n) \to \Sigma S^{n+1} \to \Sigma^2 S^n \to \cdots$$

Note that the "boundary" map $M(\mathbb{Z}_p, n)/S^n = S^{n+1} \to \Sigma S^n$ is the degree p map. We can use our standard identities to simplify

$$S^n \to M(\mathbb{Z}_p, n) \to S^{n+1} \xrightarrow{p} S^{n+1} \to M(\mathbb{Z}_p, n+1) \to S^{n+2} \to S^{n+2} \to \cdots$$

We can apply $[-, X]$ to this sequence, this will reverse the arrows and give an exact sequence

$$\pi_n(X) \leftarrow \pi_{n+1}(X; \mathbb{Z}_p) \leftarrow \pi_{n+1}(X) \xleftarrow{\times p} \pi_{n+1}(X) \leftarrow \pi_{n+1}(\mathbb{Z}_p, X) \leftarrow \cdots$$

which by looking at the $n = 1$ case can be extended to the π_1 exactly to the left.

Remark. Recall that cohomology must be represented by some spectrum and degree wise it is given by the Eilenberg-Maclane spaces

$$H^n(X;G) \cong [X, K(G,n)]$$

So *cohomology* is maps from a space to a space with a single non-zero *homotopy* group. On the other hand because S^n has a single non-zero (reduced) cohomology group in degree n we see that *homotopy* groups are maps from a space with a single non-zero *cohomology* group. The first statement works for all G however. The Moore spaces for $M(\mathbb{Z}_p, n)$ have non-zero $n-1$th cohomology and hence form the analogue of Eilenberg-maclane spaces for $n-1$, this is the reason for the shift in the notation of the $\pi_n(X;G)$, as this makes the dual statement work for at least $\mathbb{Z}_p$ coefficients.

An example of Milnor

In this section for some reason I denoted $\pi_n(G,X) = \pi_{n-1}(X,;G)$. There is a sequence (a priori) that is split short exact (a posteriori)

$$\begin{array}{ccccc}
\pi_7 O & \longrightarrow & \pi_7 PL & \longrightarrow & \pi_7 PL/O \\
\downarrow \sim & & \downarrow \sim & & \cong \\
 & & & & \Theta_7 \\
 & & & & \cong \\
\mathbb{Z} & \xrightarrow{(7,1)} & \mathbb{Z} \oplus \mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_{28}
\end{array}$$

The LHS was computed by Bott, the RHS by smoothing theory, a known homotopy group of spheres and Kervaire-Milnor and the middle group by Milnor-Brumfiel. We now construct a diagram where the horrozontals are given by the Puppe sequence in the previous section and the verticles are given by taking the homotopy groups of this sequence (maybe shifted by B).

$$\begin{array}{ccccccc}
\pi_8(PL) & \longrightarrow & \pi_7(\mathbb{Z}_7, PL) & \longrightarrow & \pi_7(PL) & \xrightarrow{\times 7} & \pi_7(PL) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_8(PL/O) & \longrightarrow & \pi_7(\mathbb{Z}_7, PL/O) & \longrightarrow & \pi_7(PL/O) & \xrightarrow{\times 7} & \pi_7(PL/O) \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_8(BO) & \longrightarrow & \pi_7(\mathbb{Z}_7, BO) & \longrightarrow & \pi_7(BO) & &
\end{array}$$

And the **claim** is that the purple arrow has a non-zero image. Are the verticle arrows exact? I think yes, non-trivial, Levine KM II. they better be I need them for the proof. First we substitute the (pertinent) known values

$$\begin{array}{ccccccc}
\pi_8(PL) & \longrightarrow & \pi_7(\mathbb{Z}_7, PL) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}_4 & \xrightarrow{(7,1)} & \mathbb{Z} \oplus \mathbb{Z}_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_8(PL/O) & \longrightarrow & \pi_7(\mathbb{Z}_7, PL/O) & \longrightarrow & \mathbb{Z}_{28} & \xrightarrow{\times 7} & \mathbb{Z}_{28} \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z} & \longrightarrow & \pi_7(\mathbb{Z}_7, BO) & \longrightarrow & 0 & &
\end{array}$$

Notice that $\theta_7 \cong bP_8$ as there are no stable homotopy group of spheres in this dimension. bP_8 is cyclic with a generator that we call σ the 'Milnor sphere'. Then consider $4\sigma \in \pi_7(PL/O)$. Notice

- that it is in the kernel of the right directed map $\times 7$ and therefore comes from an element $\xi \in \pi_7(\mathbb{Z}_7, PL/O)$,
- It is in the image of the verticle map and hence lifts to a $\zeta \in \pi_7(PL)$,
- ζ is of infinite order. If it lifted to something of fine order then by the commutativity of the diagram it would map accross by the identity and then down to a non-zero element, contradicting that it was in the kernel of the horrozontal map.

Now since ζ is of infinite order and the horrozontal map is an injection it is not in the kernel. Therefore it is not in the image of the map from $\pi_7(\mathbb{Z}_7, PL)$. But by comutativity then ξ is not in the image of the verticle map $\pi_7(\mathbb{Z}_7, PL) \to \pi_7(\mathbb{Z}_7, PL/O)$, and by exactness of the columns it is therefore not in the kernel of the second verticle map. Thus the second verticle map has a non-trivial image. □

Geometrically then what we have is a non-trivial map $[M_7, BO]$ which by pulling back gives a non-trivial vector bundle over M_7. On the other hand we know that this vector bundle does not come from a non-trivial map $[M_7, PL]$, and hence as an $\mathbb{R}^n$ fiber bundle this vector bundle is trivial. But by exactness of the columns all the vector bundles will be topologically trivial because the image of the map from $\pi_7(\mathbb{Z}_7, PL)$ will be zero (it goes through two maps), so is the point merely that some non-trivial vector bundle exists?

References

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge ; New York, 2002.